Determinants

Stuff you never heard of, but could have

I have given somewhat concise proofs of the annoying determinant facts perpetrated in *Infinitesimals reloaded*.

Rule 1: All matrices have the correct dimensions for any matrix multiplications to work and any inverse I write down exists.

Rule 2: No more rules.

Basics

1.

$$\begin{aligned}\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}\begin{bmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{bmatrix}\end{aligned}$$

Proof: Multiply them out.

2.

$$\begin{aligned}\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \det(A)\ \det(D - CA^{-1}B) \\ &= \det(D)\ \det(A - BD^{-1}C)\end{aligned}$$

Proof: A corollary of 1.

3.

$$\det\left(\begin{bmatrix} A^T \\ B^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix}\right) = \det(A^tC)\det(B^TD - B^TC(A^TC)^{-1}A^TD)$$

Proof: A corollary of 2.

Good Stuff

A. The volume of the box whose adjacent edges are the columns of the matrix B is

$$\mathrm{vol}(B) = \sqrt{\det(B^T B)}$$

Proof: Let $B = \begin{bmatrix} \bar{B} & b \end{bmatrix}$ where b is the last column of B, then

$$\det(B^T B) = \det(\bar{B}\bar{B}^T)(b^T b - b^T \bar{B}(\bar{B}^T \bar{B})^{-1}\bar{B}^T b)$$

which is just the volume of the base times the height.

B. If P is the projection of the columns of C onto the plane spanned by the columns of B, that is, $P = B(B^T B)^{-1} B^T C$, then

$$\mathrm{vol}(P) = \frac{|\det(C^T B)|}{\mathrm{vol}(B)}$$

Proof:

$$\begin{aligned} \det(P^T P) &= \det(C^T B(B^T B)^{-1} B^T C) \\ &= \frac{(\det(C^T B))^2}{\det(B^T B)} \end{aligned}$$

and take the square root.

C. If the columns of $B_\perp$ are orthogonal to the columns of B, then

$$\frac{\det \begin{bmatrix} C & B \end{bmatrix}}{\mathrm{vol}(B)} = \frac{\det(C^T B_\perp)}{\mathrm{vol}(B_\perp)}$$

Proof:

$$\begin{aligned} \det \begin{bmatrix} C & B \end{bmatrix} \det \begin{bmatrix} B_\perp & B \end{bmatrix} &= \det \begin{bmatrix} C^T B_\perp & C^T B \\ 0 & B^T B \end{bmatrix} \\ &= \det(C^T B_\perp) \det(B^T B) \end{aligned}$$

D.

$$\begin{aligned}
\det(C^T B) &= \sum C_I B_I \\
\det \begin{bmatrix} C & B^* \end{bmatrix} &= \sum \varepsilon_I C_I B^*_{I'}
\end{aligned}$$

where I is the ordered p-tuple of indices $(i_1, \ldots, i_p)$ and for an $n \times p$ matrix A, A_I is the determinant of $p \times p$ sub-matrix of A whose rows are given by I.

Proof: These facts follow amazingly enough from the fact that the determinant can be used to define an inner product, $(\cdot, \cdot)$ on the vector space of p-th order alternating vectors on R^n.

Let

$$(c_1 \wedge \ldots \wedge c_p, b_1 \wedge \ldots \wedge b_p) = \det \begin{bmatrix} c_i^T b_j \end{bmatrix}$$

and extend it linearly to all alternating p vectors. The standard basis vectors $e_I = e_{i_1} \wedge \ldots \wedge e_{i_p}$ are an orthonormal basis with this inner product.

So, for $C = \begin{bmatrix} c_1 & \cdots & c_p \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & \ldots & b_p \end{bmatrix}$ and $c_1 \wedge \ldots \wedge c_p = \sum C_I e_I$ and $b_1 \wedge \ldots \wedge b_p = \sum B_I e_I$

$$\begin{aligned}
\det(C^T B) &= (c_1 \wedge \ldots \wedge c_p, b_1 \wedge \cdots \wedge b_p) \\
&= \left(\sum C_I e_I, \sum B_J e_J \right) \\
&= \sum C_I B_I
\end{aligned}$$

For the second equation

$$\begin{aligned}
\det \begin{bmatrix} C & B^* \end{bmatrix} e_1 \wedge \ldots \wedge e_n &= c_1 \wedge \ldots \wedge c_p \wedge b_1^* \wedge \ldots \wedge b_{n-p}^* \\
&= \sum C_I e_I \wedge \sum C^*_{I'} e_{I'} \\
&= \sum C_I B^*_{I'} e_I \wedge e_{I'} \\
&= \sum \varepsilon_I C_i B^*_{I'} e_1 \wedge \ldots \wedge e_n
\end{aligned}$$